Last Time: X a r.v., g a $f^{\underset{\sim}{}}$; $Y = g(X)$

Q/ How to find distribution of Y?

(ie, derived distribution)

A/ Most of the time, not needed. However, possible in a few special/important cases.

ex: $Y = aX + b$ $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

Today: A few more examples

Order Statistics

- Let $X_1, \dots, X_n$ be iid rv's. & sort them s.t.

$$\underbrace{X^{(1)} \le X^{(2)} \le \dots \le X^{(n)}}_{\text{"order statistics"}}$$

essentially just rearranging the fcns

Ex:

$$X^{(1)}(\omega) = \min\{X_1(\omega) \dots X_n(\omega)\}$$
$$X^{(n)}(\omega) = \max\{X_1(\omega) \dots X_n(\omega)\}$$

Claim: If X_i's are cts rv's w/ density f_X, then:

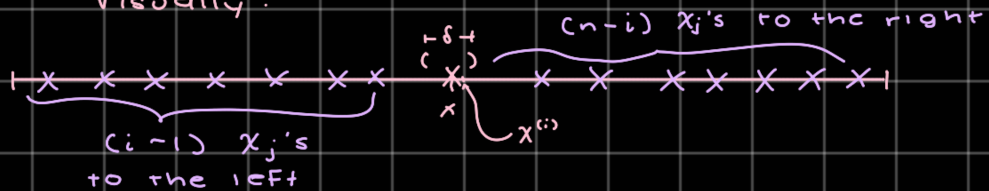
$$f_{X^{(i)}}(x) = n \binom{n-1}{i-1} F_X(x)^{(i-1)} \left(1 - F_X(x)\right)^{n-i} f_X(x)$$

- n: # of ways to select X_j that falls in $(x, x+\delta)$
- $\binom{n-1}{i-1}$: # ways to select $(i-1)$ X_j's $\le x$ from remaining $(n-1)$
- $F_X(x)^{(i-1)}$: probability that the $(i-1)$ X_j's are $\le x$
- $(1-F_X(x))^{n-i}$: Probability that $(n-i)$ X_j's on the right are $\ge x+\delta$
- $f_X(x)$: $\approx \delta P\{X_j \in (x, x+\delta)\}$ ↳ "marginal density of x's"

↳ "proof"

$$f_{X^{(i)}}(x) \approx \delta P\{X^{(i)} \in (x, x+\delta)\}$$

↳ Visually:



Numerical Examples

1) n identical batteries with lifespans $X_i \sim \text{Exp}(\lambda)$

- Time to 1st dead battery: $X^{(1)}$

$$f_{X^{(1)}}(x) = n\binom{n-1}{1-1}(1-e^{-\lambda x})^{1-1}(e^{-\lambda x})^{n-1}\lambda e^{-\lambda x}$$

$$= \underbrace{n\lambda e^{-n\lambda x}}$$

$\Rightarrow X^{(1)} \sim \text{Exp}(n\lambda)$

"is exponential with rate $n\lambda$"

- Time to last dead battery:

$$f_{X^{(n)}}(x) = n\binom{n-1}{n-1}(1-e^{-\lambda x})^{n-1}(e^{-\lambda x})^{n-n}\lambda e^{-\lambda x}$$

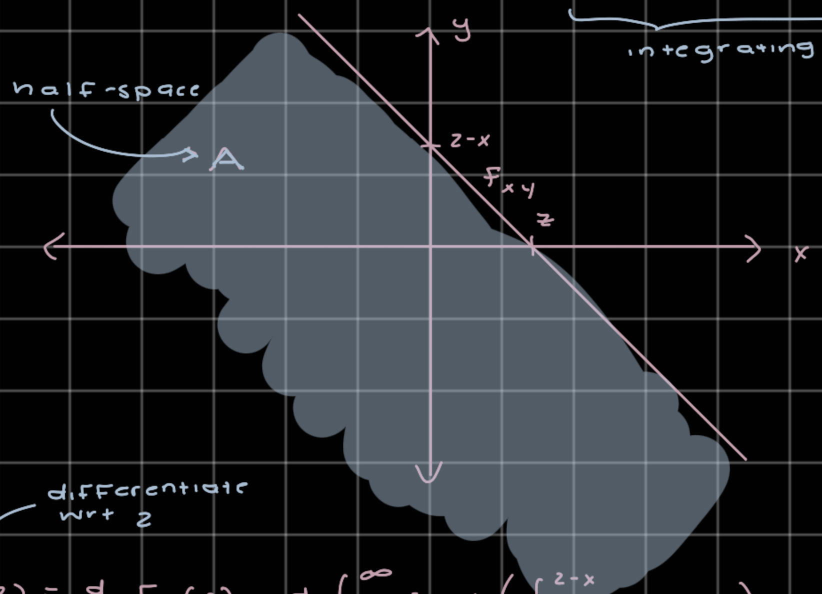
$$= \underbrace{\lambda n(1-e^{-\lambda x})^{n-1}e^{-\lambda x}}_{\text{not exponential}}$$

2) Another example of derived distribution we can find explicitly is that of the sum of independent rv's.

Claim: If X, Y are cts rv's, independent then $Z = X+Y$ has density

$$f_Z(z) = (f_X \cdot f_Y)(z) := \int f_X(x)\, f_Y(z-x)\, dx$$

$$F_Z(z) = P\{X+Y \leq z\} = \underbrace{\int\int_{-\infty}^{z-x} f_{XY}(x,y)\, dy\, dx}_{\text{integrating over } A}$$



differentiate wrt z

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{d}{dz}\int_{-\infty}^{\infty} f_X(x)\left(\int_{-\infty}^{z-x} f_Y(y)\, dy\right) dx$$

iterated integration

$$= \int_{-\infty}^{\infty} f_x(x) \underbrace{\frac{d}{dz}\left(\int_{-\infty}^{\infty} f_y(y)\,dy\right)}_{f_y(z-x)} dx$$

3) X, Y discrete, integer-valued, independent

$$P_Z(n) = P\{Z=n\} \underset{\text{Law of total Prob}}{=} \sum_{k\in\mathbb{Z}} P(Z=n \mid X=k)\,P_x(k)$$

$$= \sum_{k\in\mathbb{Z}} P(X+Y=n \mid X=k)\,P_x(k)$$

$$= \sum_{k\in\mathbb{Z}} P(Y=n-k \mid X=k)\,P_x(k)$$

$$= \sum_{k} P_x(k)\,P_y(n-k) = \underbrace{(P_x * P_y)}_{\text{convolution}}(n)$$

4) $X \sim \mathcal{N}(0, \sigma_x^2)$ indep of $Y \sim \mathcal{N}(0, \sigma_y^2)$ $\quad Z = X+Y$

$$f_z(z) = \frac{1}{\sqrt{2\pi\sigma_x^2}\sqrt{2\pi\sigma_y^2}} \int e^{-\frac{x^2}{2\sigma_x^2}} e^{-\frac{(z-x)^2}{2\sigma_y^2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}\sqrt{2\pi\sigma_y^2}} \exp\left\{\frac{-z^2}{2(\sigma_x^2+\sigma_y^2)}\right\} \underbrace{\int \exp\left(-\frac{1}{\left(\frac{2\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_y^2}\right)}\left(x - \frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2} z\right)^2\right) dx}_{\text{Gaussian density}}$$

$$\sqrt{\frac{2\pi\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_y^2}}$$

$$= \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}} \exp\left\{\frac{-z^2}{2(\sigma_x^2+\sigma_y^2)}\right\}$$

Moment-Generating Functions

↳ essentially a transform of the distribution

- **def:** For a r.v. X define its MGF as:

$$M_X(t) = \mathbb{E}[e^{tX}], \quad t \in \mathbb{R}$$

↳ for "nice" distributions, M_X uniquely characterizes

↳ $t=0 \Rightarrow \mathbb{E}=1$, $t>0 \Rightarrow \mathbb{E}\to\infty$ (divergence) (need to avoid this case) the distribution of X.

Ex:

$X \sim \mathcal{N}(\mu, \sigma^2) \qquad M_x(t) = \exp\{\mu t + \frac{\sigma^2 t^2}{2}\}$

- Suppose you compute for rv Y, you get

$$M_y(t) = \exp\{3t + \frac{t^2}{2}\}$$
$$\Rightarrow Y \sim \mathcal{N}(3, 1)$$

Ex Suppose X, Y indep

$$\begin{aligned} M_{x+y}(t) &= \mathbb{E}[e^{t(x+y)}] \\ &= \mathbb{E}[e^{tx} e^{ty}] \\ &= \mathbb{E}[e^{tx}]\,\mathbb{E}[e^{ty}] \\ &= M_x(t) M_y(t) \end{aligned}$$

bc independent

Ex $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$, $Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$ $\quad Z = X + Y$

indep

$$M_Z(t) = M_x(t) M_y(t) = \exp\{\mu_x t + \frac{\sigma_x^2 t^2}{2}\} \exp\{\mu_y t + \frac{\sigma_y^2 t^2}{2}\}$$

$$= \exp\left((\mu_x + \mu_y)t + \frac{(\sigma_x^2 + \sigma_y^2)t^2}{2}\right)$$

MGF of a Gaussian

$$\Rightarrow Z \sim \mathcal{N}(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

Q/ Why is it called a "moment generating fcn"?

$$e^x = \sum_{k \geq 0} \frac{x^k}{k!}$$

$$\mathbb{E}[e^{tx}] = \sum_{k \geq 0} \underbrace{\frac{t^k}{k!}}_{\text{coeff of polynomial in } t} \mathbb{E}[x^k]$$

$$\frac{d^k}{dt^k} M_x(t) \Big|_{t=0} = \underbrace{\mathbb{E}[x^k]}_{k^{th} \text{ moment}}$$

diff. k times

Concentration Inequalities

$$P\{X \in B\} \approx 0 \quad \text{or} \quad P\{X \in B\} \approx 1$$

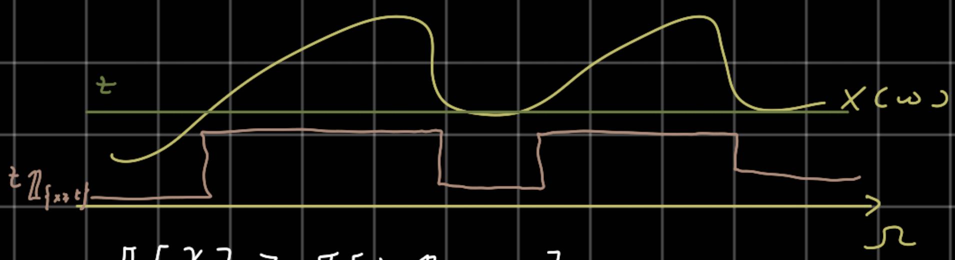
Markov's Inequality

↳ relates through mean

If X non-neg rv, then:

$$P\{X \geq t\} \leq \frac{\mathbb{E}X}{t}$$

Visually:



$$\mathbb{E}[X] \geq \mathbb{E}[t\,\mathbb{1}_{\{X \geq t\}}]$$
$$= t\,P\{X \geq t\}$$

Example:

$$P\{|X - \mathbb{E}X| \geq t\} \leq \frac{\mathbb{E}|X - \mathbb{E}[X]|^k}{t^k} \qquad \forall t > 0$$

$$P\{|X - \mathbb{E}[X]|^k \geq t^k\} \underbrace{\leq}_{\text{Markov}} \frac{\mathbb{E}|X - \mathbb{E}[X]|^k}{t^k}$$